

# SOLVING THE HEAT EQUATION USING NONSTANDARD ANALYSIS

TRISTRAM DE PIRO

ABSTRACT. We use the nonstandard Fourier transform method, see [6], along with an established nonstandard approach to ODE's, see [2] and [7], to find a solution to the heat equation, on  $(0, \infty) \times \mathcal{R}$ , with a given boundary condition  $g$  at  $t = 0$ . We use this result to find an algorithm, converging to a solution of this equation, with applications to derivatives pricing in finance.

We adopt the following notation;

**Definition 0.1.** For  $\eta \in {}^*\mathcal{N} \setminus \mathcal{N}$ , we let  $(\overline{\mathcal{R}}_\eta, \mathfrak{C}_\eta, \lambda_\eta)$  be as in Definition 0.15 of [6].

We let  $(\overline{\mathcal{R}}_\eta, L(\mathfrak{C}_\eta), L(\lambda_\eta))$  denote the associated Loeb space, see Definition 0.5 of [6].

$(\mathcal{R}, \mathfrak{B}, \mu), (\mathcal{R}^{+-\infty}, \mathfrak{B}', \mu')$  are as in Lemma 0.6 of [6].

$\overline{\mathcal{T}}_\eta = \{\tau \in {}^*\mathcal{R}_{\geq 0} : 0 \leq \tau < \eta\}$  and we again denote by  $\mathfrak{C}_\eta$ , the restriction of  $\mathfrak{C}_\eta$  to  $\overline{\mathcal{T}}_\eta$ , and  $\lambda_\eta$  the restriction of the counting measure.

$(\overline{\mathcal{T}}_\eta, L(\mathfrak{C}_\eta), L(\lambda_\eta))$  is the corresponding Loeb space.

$\mathcal{T} = \mathcal{R}_{\geq 0}$  and  $(\mathcal{T}, \mathfrak{B}, \mu), (\mathcal{T}^{+\infty}, \mathfrak{B}', \mu')$  are defined analogously to Lemma 0.6 of [6].

$(\overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta, \mathfrak{C}_\eta^2, \lambda_\eta^2)$  is as in Definition 0.15 of [6].

$(\overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta, L(\mathfrak{C}_\eta^2), L(\lambda_\eta^2))$  is the corresponding Loeb space.

$(\overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta, L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta), L(\lambda_\eta) \times L(\lambda_\eta))$  is the complete product of the Loeb spaces  $(\overline{\mathcal{T}}_\eta, L(\mathfrak{C}_\eta), L(\lambda_\eta))$  and  $(\overline{\mathcal{R}}_\eta, L(\mathfrak{C}_\eta), L(\lambda_\eta))$ .

Similarly,  $(\mathcal{T}^{+\infty} \times \mathcal{R}^{+-\infty}, \mathfrak{B}' \times \mathfrak{B}', \mu' \times \mu')$  and  $(\mathcal{T} \times \mathcal{R}, \mathfrak{B} \times \mathfrak{B}, \mu \times \mu)$  are the complete products of  $(\mathcal{T}^{+\infty}, \mathfrak{B}', \mu')$ ,  $(\mathcal{R}^{+-\infty}, \mathfrak{B}', \mu')$  and  $(\mathcal{T}, \mathfrak{B}, \mu)$ ,  $(\mathcal{R}, \mathfrak{B}, \mu)$  respectively.

We let  $({}^*\mathcal{R}, {}^*\mathfrak{D})$  denote the hyperreals, with the transfer of the Borel field  $\mathfrak{D}$  on  $\mathcal{R}$ . A function  $f : (\overline{\mathcal{R}_\eta}, \mathfrak{C}_\eta) \rightarrow ({}^*\mathcal{R}, {}^*\mathfrak{D})$  is measurable, if  $f^{-1} : {}^*\mathfrak{D} \rightarrow \mathfrak{C}_\eta$ . Similarly,  $f : (\overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta}, \mathfrak{C}_\eta^2) \rightarrow ({}^*\mathcal{R}, {}^*\mathfrak{D})$  is measurable, if  $f^{-1} : {}^*\mathfrak{D} \rightarrow \mathfrak{C}_\eta^2$ . Observe that this is equivalent to the definition given in [4]. We will abbreviate this notation to  $f : \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{R}$  or  $f : \overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{R}$  is measurable,  $(*)$ . The same applies to  $({}^*\mathcal{C}, {}^*\mathfrak{D})$ , the hyper complex numbers, with the transfer of the Borel field  $\mathfrak{D}$ , generated by the complex topology. Observe that  $f : \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$  or  $f : \overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$  is measurable, in this sense, iff  $\text{Re}(f)$  and  $\text{Im}(f)$  are measurable in the sense of  $(*)$ .

We have the following lemma, generalising Theorem 0.7 of [6] and Theorem 22 of [1];

**Lemma 0.2.** *The identity;*

$$i : (\overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta}, L(\mathfrak{C}_\eta^2), L(\lambda_\eta^2)) \rightarrow (\overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta}, L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta), L(\lambda_\eta) \times L(\lambda_\eta))$$

*and the standard part mapping;*

$$\begin{aligned} st : (\overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta}, L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta), L(\lambda_\eta) \times L(\lambda_\eta)) \\ \rightarrow (\mathcal{T}^{+\infty} \times \mathcal{R}^{+-\infty}, \mathfrak{B}' \times \mathfrak{B}', \mu' \times \mu') \end{aligned}$$

*are measurable and measure preserving.*

*Proof.* To show that  $i$  is measurable and measure preserving, it is sufficient to prove that;

- (i).  $L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta) \subset L(\mathfrak{C}_\eta^2)$ .
- (ii).  $L(\lambda_\eta^2)|_{L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta)} = L(\lambda_\eta) \times L(\lambda_\eta)$ .

As in [1], if  $A \in \mathfrak{C}_\eta$ ;

$$\{M \in \sigma(\mathfrak{C}_\eta) : M \times A \in \sigma(\mathfrak{C}_\eta \times \mathfrak{C}_\eta)\}$$

is a  $\sigma$ -algebra, containing  $\mathfrak{C}_\eta$ , hence, it equals  $\sigma(\mathfrak{C}_\eta)$ . Similarly, if  $B \in \sigma(\mathfrak{C}_\eta)$ ;

$$\{M \in \sigma(\mathfrak{C}_\eta) : B \times M \in \sigma(\mathfrak{C}_\eta \times \mathfrak{C}_\eta)\}$$

is a  $\sigma$ -algebra, and equals  $\sigma(\mathfrak{C}_\eta)$ . Therefore;

$$\mathfrak{C}_\eta \times \mathfrak{C}_\eta \subset \sigma(\mathfrak{C}_\eta) \times \sigma(\mathfrak{C}_\eta) = \sigma(\mathfrak{C}_\eta \times \mathfrak{C}_\eta)$$

Now, using Ward Henson's result, see footnote 1 of [6], it follows that  $L(\lambda_\eta^2) = L(\lambda_\eta) \times L(\lambda_\eta)$  on  $\sigma(\mathfrak{C}_\eta) \times \sigma(\mathfrak{C}_\eta)$ ,  $(*)$ . Now, suppose that  $\{C, D\} \subset L(\mathfrak{C}_\eta)$  then, there exists  $\{C_1, C_2, D_1, D_2\} \subset \sigma(\mathfrak{C}_\eta)$ , with  $C_1 \subset C \subset C_2$ ,  $D_1 \subset D \subset D_2$ ,  $L(\lambda_\eta)(C_2 \setminus C_1) = 0$ ,  $L(\lambda_\eta)(D_2 \setminus D_1) = 0$ ,  $(**)$ , and  $C_1 \times D_1 \subset C \times D \subset C_2 \times D_2$ . Moreover,  $(C_2 \times D_2 \setminus C_1 \times D_1) \subset ((C_2 \setminus C_1) \times D_2) \cup (C_2 \times (D_2 \setminus D_1))$ ,  $(*** )$ . By  $(*)$ ,  $(**)$ ,  $(*** )$ ,  $L(\lambda_\eta^2)(C_2 \times D_2 \setminus C_1 \times D_1) = 0$ . Therefore,  $C \times D \in L(\mathfrak{C}_\eta^2)$ , and the product  $\sigma$ -algebra  $L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta) \subset L(\mathfrak{C}_\eta^2)$ ,  $(\dagger)$ . Using  $(*)$ ,  $(\dagger)$ ,  $L(\lambda_\eta^2)$  agrees with  $L(\lambda_\eta) \times L(\lambda_\eta)$  on this algebra, hence, the complete product  $L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta) \subset L(\mathfrak{C}_\eta^2)$ , showing  $(i)$ , and  $L(\lambda_\eta^2)|_{L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta)} = L(\lambda_\eta) \times L(\lambda_\eta)$ , by the definition of a completion, showing  $(ii)$ .

We recall the result, Theorem 0.7, of [6], that;

$$st : (\overline{\mathcal{R}_\eta}, L(\mathfrak{C}_\eta), L(\lambda_\eta)) \rightarrow (\mathcal{R}^{+-\infty}, \mathfrak{B}', \mu')$$

is measurable and measure preserving,  $(\sharp)$ . Similarly, one can show that;

$$st : (\overline{\mathcal{T}_\eta}, L(\mathfrak{C}_\eta), L(\lambda_\eta)) \rightarrow (\mathcal{T}^{+\infty}, \mathfrak{B}', \mu')$$

is measurable and measure preserving,  $(\sharp\sharp)$ . The rest of the argument is fairly straightforward, if  $\{B_1, B_2\} \subset \mathfrak{B}'$ , then, using  $(\sharp)$ ,  $(\sharp\sharp)$ ,  $st^{-1}(B_1 \times B_2) \in L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta)$ , and  $L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta)(st^{-1}(B_1 \times B_2)) = \mu' \times \mu'(B_1 \times B_2)$ . It follows, using the usual argument, as in the first part of the proof, that the push forward measure  $st_*(L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta))$  agrees with  $\mu' \times \mu'$  on  $\mathfrak{B}' \times \mathfrak{B}'$ , considered as a product  $\sigma$ -algebra. Then, the result follows easily from the definition of a complete product.  $\square$

The following definition is based on Definition 0.18 of [6];

**Definition 0.3.** *Discrete Partial Derivatives*

Let  $f : \overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta \rightarrow {}^*\mathcal{C}$  be measurable. Then we define  $\frac{\partial f}{\partial t}$  to be the unique measurable function satisfying;

$$\frac{\partial f}{\partial t}(\frac{j}{\eta}, x) = \eta(f(\frac{j+1}{\eta}, x) - f(\frac{j}{\eta}, x)) \text{ for } j \in {}^*\mathcal{N}_{0 \leq j \leq \eta^2-2}, x \in \overline{\mathcal{R}}_\eta$$

$$\frac{\partial f}{\partial t}(\frac{\eta^2-1}{\eta}, x) = 0$$

$$\frac{\partial f}{\partial x}(t, \frac{j}{\eta}) = \eta(f(t, \frac{j+1}{\eta}) - f(t, \frac{j}{\eta})) \text{ for } j \in {}^*\mathcal{N}_{-\eta^2 \leq j \leq \eta^2-2}, t \in \overline{\mathcal{T}}_\eta$$

$$\frac{\partial f}{\partial x}(t, \frac{\eta^2-1}{\eta}) = 0$$

**Remarks 0.4.** If  $f$  is measurable, then so are  $\frac{\partial f}{\partial t}$ ,  $\frac{\partial f}{\partial x}$  and  $\frac{\partial^2 f}{\partial x^2}$ . This follows immediately, by transfer, from the corresponding result for the discrete derivatives of discrete functions  $f : \mathcal{T}_n \times \mathcal{R}_n \rightarrow \mathcal{C}$ , where  $n \in \mathcal{N}$ , see Definition 0.15 and Definition 0.18 of [6].

**Lemma 0.5.** Given a measurable boundary condition  $g : \overline{\mathcal{R}}_\eta \rightarrow {}^*\mathcal{C}$ , there exists a unique measurable  $f : \overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta \rightarrow {}^*\mathcal{C}$ , satisfying the non-standard heat equation;

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = 0 \text{ on } (\overline{\mathcal{T}}_\eta \setminus [\frac{\eta^2-1}{\eta}, \eta)) \times \overline{\mathcal{R}}_\eta$$

with  $f(0, x) = g(x)$ , for  $x \in \overline{\mathcal{R}}_\eta$ , (\*).

*Proof.* Observe that, by Definition 0.3, if  $f : \overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta \rightarrow {}^*\mathcal{C}$  is measurable, then;

$$\frac{\partial^2 f}{\partial x^2}(t, \frac{j}{\eta}) = \eta^2(f(t, \frac{j+2}{\eta}) - 2f(t, \frac{j+1}{\eta}) + f(t, \frac{j}{\eta})), (-\eta^2 \leq j \leq \eta^2 - 3).$$

$$\frac{\partial^2 f}{\partial x^2}(t, \frac{\eta^2-2}{\eta}) = -\eta^2(f(t, \frac{\eta^2-1}{\eta}) - f(t, \frac{\eta^2-2}{\eta}))$$

$$\frac{\partial^2 f}{\partial x^2}(t, \frac{\eta^2-1}{\eta}) = 0$$

Therefore, if  $f$  satisfies (\*), we must have;

$$f(\frac{i+1}{\eta}, \frac{j}{\eta}) = f(\frac{i}{\eta}, \frac{j}{\eta}) + \eta(f(\frac{i}{\eta}, \frac{j+2}{\eta}) - 2f(\frac{i}{\eta}, \frac{j+1}{\eta}) + f(\frac{i}{\eta}, \frac{j}{\eta})),$$

$$(0 \leq i \leq \eta^2 - 2, -\eta^2 \leq j \leq \eta^2 - 3).$$

$$f\left(\frac{i+1}{\eta}, \frac{\eta^2-2}{\eta}\right) = f\left(\frac{i}{\eta}, \frac{\eta^2-2}{\eta}\right) - \eta\left(f\left(\frac{i}{\eta}, \frac{\eta^2-1}{\eta}\right) - f\left(\frac{i}{\eta}, \frac{\eta^2-2}{\eta}\right)\right),$$

$$(0 \leq i \leq \eta^2 - 2).$$

$$f\left(\frac{i+1}{\eta}, \frac{\eta^2-1}{\eta}\right) = f\left(\frac{i}{\eta}, \frac{\eta^2-1}{\eta}\right), (0 \leq i \leq \eta^2 - 2).$$

$$f\left(0, \frac{j}{\eta}\right) = g\left(\frac{j}{\eta}\right), (-\eta^2 \leq j \leq \eta^2 - 1). (**)$$

If  $\eta = n \in \mathcal{N}$ , then given any measurable  $g : \mathcal{R}_n \rightarrow \mathcal{C}$ , the condition (\*\*), clearly determines a unique measurable, see Definition 0.15 of [6],  $f : \mathcal{T}_n \times \mathcal{R}_n \rightarrow \mathcal{C}$ , satisfying (\*). As the condition (\*) can be written down uniformly, in Robinson's higher order logic, we obtain the result, immediately, by transfer.  $\square$

**Definition 0.6.** We recall the definition from [6], Definition 0.15. Given a measurable  $f : \overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$ , we define  $\exp_\eta(-\pi ixy)$  and  $\exp(\pi ixy)$  to be the  $\mathfrak{C}_\eta^2$  measurable counterparts of the transfers of  $\exp(\pi ixy)$  and  $\exp(-\pi ixy)$  to  $\overline{\mathcal{R}_\eta}^2$ . We define the nonstandard Fourier transform in space;

$$\hat{f}(t, y) = \int_{\overline{\mathcal{R}_\eta}} f(t, x) \exp_\eta(-\pi ixy) d\lambda_\eta(x)$$

and the nonstandard inverse Fourier transform in space;

$$\check{f}(t, y) = \int_{\overline{\mathcal{R}_\eta}} f(t, x) \exp_\eta(\pi ixy) d\lambda_\eta(x)$$

As in Definition 0.20 of [6], we let  $\phi_\eta, \psi_\eta : \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$  be defined by;

$$\phi_\eta(x) = \eta(\exp_\eta(-\pi i\frac{x}{\eta}) - 1)$$

$$\psi_\eta(x) = \eta(\exp_\eta(\pi i\frac{x}{\eta}) - 1)$$

If  $f$  is measurable, we let;

$$C_\eta(t, x) = f\left(t, \frac{\eta^2-1}{\eta}\right) \exp_\eta(-\pi i\frac{\eta^2-1}{\eta}x) - f(t, -\eta) \exp_\eta(-\pi i(-\eta)x)$$

$$D_\eta(t, x) = -\frac{1}{\eta} f(t, -\eta) \exp_\eta(\pi i\frac{x}{\eta}) \exp_\eta(-\pi i(-\eta)x).$$

$$C'_\eta(t, x) = -\frac{\partial f}{\partial x}(t, -\eta) \exp_\eta(-\pi i(-\eta)x)$$

$$D'_\eta(t, x) = -\frac{1}{\eta} \frac{\partial f}{\partial x}(t, -\eta) \exp_\eta(\pi i \frac{x}{\eta}) \exp_\eta(-\pi i(-\eta)x).$$

$$E_\eta(t, x) = \phi_\eta(x) D_\eta(t, x) - C_\eta(t, x)$$

$$E'_\eta(t, x) = \phi_\eta(x) D'_\eta(t, x) - C'_\eta(t, x)$$

$$F_\eta(t, x) = \psi_\eta(x) \phi_\eta(x) D_\eta(t, x) - \psi_\eta(x) C_\eta(t, x) + \phi_\eta(x) D'_\eta(t, x) - C'_\eta(t, x)$$

**Remarks 0.7.** *If  $f$  is measurable, then so are  $\hat{f}$  and  $\check{f}$ . Again this follows, by transfer, from the finite case, as in Remark 0.4. By Lemma 0.16 of [6], if  $f$  is measurable, then, we have the nonstandard inversion theorems;*

$$\check{f} = 2f$$

$$\hat{f} = 2f$$

**Lemma 0.8.** *If  $f : \overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta \rightarrow {}^*\mathcal{C}$  is measurable, then;*

$$(i). \quad \frac{\partial \hat{f}}{\partial t} = \frac{\partial f}{\partial t}.$$

$$(ii). \quad \frac{\partial^2 \hat{f}}{\partial x^2} = \psi_\eta^2 \hat{f} - F_\eta.$$

*Proof.* (i). Using Definition 0.3 and Definition 0.6, we have:

$$\begin{aligned} \frac{\partial \hat{f}}{\partial t}(t', y) &= \int_{\overline{\mathcal{R}}_\eta} \frac{\partial f}{\partial t}(t', x) \exp_\eta(-\pi i x y) d\lambda_\eta(x) \\ &= \eta \left( \int_{\overline{\mathcal{R}}_\eta} f(t' + \frac{1}{\eta}, x) \exp_\eta(-\pi i x y) d\lambda_\eta(x) - \int_{\overline{\mathcal{R}}_\eta} f(t', x) \exp_\eta(-\pi i x y) d\lambda_\eta(x) \right), \\ &\hspace{25em} (0 \leq t' < \frac{\eta^2-1}{\eta}) \\ &= \eta(\hat{f}(t' + \frac{1}{\eta}, y) - \hat{f}(t', y)) = \frac{\partial \hat{f}}{\partial t}(t', y), \quad (0 \leq t' < \frac{\eta^2-1}{\eta}) \\ \frac{\partial \hat{f}}{\partial t}(t', y) &= \frac{\partial \hat{f}}{\partial t}(t', y) = 0, \quad (\frac{\eta^2-1}{\eta} \leq t' < \eta) \end{aligned}$$

(ii). Using the definition of  $\psi_\eta$  and  $F_\eta$  in Definition 0.6, and the transfer of the result in Lemma 0.21 of [6].

□

**Theorem 0.9.** *Let  $f : \overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta \rightarrow {}^*\mathcal{C}$  satisfy the conditions of Lemma 0.5. Then  $\hat{f}$  is determined by;*

$$\hat{f}\left(\frac{i}{\eta}, x\right) = \hat{g}(x)\left(1 + \frac{\psi_\eta^2(x)}{\eta}\right)^i - \frac{1}{\eta} {}^*\sum_{0 \leq j \leq i-1} F_\eta\left(\frac{j}{\eta}, x\right)\left(1 + \frac{\psi_\eta^2(x)}{\eta}\right)^{i-j-1},$$

$$(0 \leq i \leq \eta^2 - 1), (*)$$

*In particular, if the boundary condition  $g$  satisfies;*

$$g\left(\frac{\eta^2-1}{\eta}\right) = 0$$

$$g(x) = 0, \text{ for } -\eta \leq x < -\eta + \omega, \text{ where } \omega \in {}^*\mathcal{N} \setminus \mathcal{N}$$

*then,  $\hat{f}$  is determined by;*

$$\hat{f}\left(\frac{i}{\eta}, x\right) = \hat{g}(x)\left(1 + \frac{\psi_\eta^2(x)}{\eta}\right)^i, (0 \leq i \leq n\eta, n \in \mathcal{N})$$

*Proof.* We have that;

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = 0 \text{ on } (\overline{\mathcal{T}}_\eta \setminus [\frac{\eta^2-1}{\eta}, \eta)) \times \overline{\mathcal{R}}_\eta$$

Applying the nonstandard Fourier transform, and using Lemma 0.8, we have;

$$\frac{\partial \hat{f}}{\partial t} - (\psi_\eta^2 \hat{f} - F_\eta) = 0 \text{ on } (\overline{\mathcal{T}}_\eta \setminus [\frac{\eta^2-1}{\eta}, \eta)) \times \overline{\mathcal{R}}_\eta$$

Using Definition 0.3, we have;

$$\eta(\hat{f}(\frac{k+1}{\eta}, x) - \hat{f}(\frac{k}{\eta}, x)) = \psi_\eta^2(x)\hat{f}(\frac{k}{\eta}, x) - F_\eta(\frac{k}{\eta}, x)$$

$$\hat{f}(\frac{k+1}{\eta}, x) = \hat{f}(\frac{k}{\eta}, x)\left(1 + \frac{\psi_\eta^2(x)}{\eta}\right) - \frac{1}{\eta} F_\eta(\frac{k}{\eta}, x), (0 \leq k \leq \eta^2 - 2). (**)$$

Let  $A = \{i \in {}^*\mathcal{N} : 0 \leq i \leq \eta^2 - 1, \text{ for which } (*) \text{ holds}\}$ . Then  $A$  is internal,  $A(0)$  holds, as  $\hat{f}(0, x) = \hat{g}(x)$ , by the boundary condition in Lemma 0.5, and if  $A(i)$  holds, for  $0 \leq i \leq \eta^2 - 2$ , then, using (\*\*);

$$\hat{f}\left(\frac{i+1}{\eta}, x\right) = [\hat{g}(x)\left(1 + \frac{\psi_\eta^2(x)}{\eta}\right)^i$$

$$- \frac{1}{\eta} {}^*\sum_{0 \leq j \leq i-1} F_\eta\left(\frac{j}{\eta}, x\right)\left(1 + \frac{\psi_\eta^2(x)}{\eta}\right)^{i-j-1}]\left(1 + \frac{\psi_\eta^2(x)}{\eta}\right) - \frac{1}{\eta} F_\eta\left(\frac{i}{\eta}, x\right)$$

$$\begin{aligned}
&= \hat{g}(x)(1 + \frac{\psi_\eta^2(x)}{\eta})^{i+1} - \frac{1}{\eta} \sum_{0 \leq j \leq i-1} F_\eta(\frac{i}{\eta}, x)(1 + \frac{\psi_\eta^2(x)}{\eta})^{i-j} - \frac{1}{\eta} F_\eta(\frac{i}{\eta}, x) \\
&= \hat{g}(x)(1 + \frac{\psi_\eta^2(x)}{\eta})^{i+1} - \frac{1}{\eta} \sum_{0 \leq j \leq i} F_\eta(\frac{i}{\eta}, x)(1 + \frac{\psi_\eta^2(x)}{\eta})^{i-j}
\end{aligned}$$

so  $A(i+1)$  holds. It follows, by hyperfinite induction, see [7], that  $A = \{i \in {}^*\mathcal{N} : 0 \leq i \leq \eta^2 - 1\}$ , and  $\hat{f}$  is determined by the condition (\*).

Now suppose that the boundary condition  $g$  satisfies the requirements in the second part of the Theorem, then, using Lemma 0.5, we have;

$$f(\frac{i}{\eta}, \frac{\eta^2-1}{\eta}) = f(0, \frac{\eta^2-1}{\eta}) = g(\frac{\eta^2-1}{\eta}) = 0, \quad (0 \leq i \leq \eta^2 - 1)$$

Moreover, again by Lemma 0.5,  $f(\frac{i}{\eta}, \frac{-\eta^2+1}{\eta})$  and  $f(\frac{i}{\eta}, -\eta)$  are hyperfinite linear combinations of the values  $g(\frac{j}{\eta})$ , for  $-\eta^2 \leq j \leq -\eta^2+1+2i$ . For such  $j$ , and  $0 \leq i \leq n\eta$ ,  $\frac{j}{\eta} \leq -\eta + \frac{1+2i}{\eta} \leq -\eta + 1 + 2n < -\eta + \omega$ , so  $g(\frac{j}{\eta}) = 0$  by hypothesis, and, then,  $f(\frac{i}{\eta}, \frac{-\eta^2+1}{\eta}) = f(\frac{i}{\eta}, -\eta) = 0$ , for  $0 \leq i \leq n\eta$ . Checking the Definition 0.6, it follows that  $F_\eta(\frac{i}{\eta}, x) = 0$ , for  $0 \leq i \leq n\eta$ ,  $n \in \mathcal{N}$ . Then, using the first part of the Theorem, we obtain the final result.  $\square$

**Definition 0.10.** *Convolution*

Suppose that  $f, g : \overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$  are measurable. Then we define the nonstandard convolution by;

$$(f * g)(t, x) = \int_{\overline{\mathcal{R}_\eta}} f(t, \frac{[ \eta x ]}{\eta} - y) g(t, y) d\nu_\eta(y)$$

**Theorem 0.11.** *Nonstandard Convolution Theorem*

Let hypotheses be as in 0.10, then;

$$f \hat{*} g = \hat{f} \hat{g} \quad f \check{*} g = \check{f} \check{g}$$

*Proof.* This is a straightforward computation. We have, for  $x \in \overline{\mathcal{R}_\eta}$ , using Definition 0.15 of [6], that;

$$\hat{f} \hat{g}(t, x) = \frac{1}{\eta^2} [{}^*\sum_{j=-\eta^2}^{\eta^2-1} f(t, \frac{j}{\eta}) \exp_\eta(-\pi i(\frac{j}{\eta})x)] [{}^*\sum_{k=-\eta^2}^{\eta^2-1} g(t, \frac{k}{\eta}) \exp_\eta(-\pi i(\frac{k}{\eta})x)]$$



$$\begin{aligned}
&= \frac{1}{\eta^2} * \sum_{j,k=-\eta^2}^{\eta^2-1} f(t, \frac{j}{\eta}) g(t, \frac{k}{\eta}) \exp_{\eta}(-\pi i(\frac{j+k}{\eta})x) \\
&= \frac{1}{\eta^2} * \sum_{l,k=-\eta^2}^{\eta^2-1} f(t, \frac{l-k}{\eta}) g(t, \frac{k}{\eta}) \exp_{\eta}(-\pi i(\frac{l}{\eta})x) \quad (l = j + k) \\
&= \frac{1}{\eta} * \sum_{l=-\eta^2}^{\eta^2-1} (\int_{\overline{\mathcal{R}_{\eta}}} f(t, \frac{l}{\eta} - w) g(w) d\nu_{\eta}(w)) \exp_{\eta}(-\pi i(\frac{l}{\eta})x) \\
&= \frac{1}{\eta} * \sum_{l=-\eta^2}^{\eta^2-1} (f * g)(t, \frac{l}{\eta}) \exp_{\eta}(-\pi i(\frac{l}{\eta})x) \\
&= f \hat{*} g(t, x)
\end{aligned}$$

A similar calculation shows that  $f \check{*} g = \check{f} \check{g}$

□

**Definition 0.12.** For  $\omega' \in {}^*\mathcal{N} \setminus \mathcal{N}$ , with  $\omega' < \eta$ , we let  $F_{\omega'} : \overline{\mathcal{T}_{\eta}} \times \overline{\mathcal{R}_{\eta}} \rightarrow {}^*\mathcal{R}$  be the measurable function defined by;

$$F_{\omega'}(t, \frac{j}{\eta}) = \frac{1}{2}, \text{ if } -\omega'\eta \leq j \leq \omega'\eta$$

$$F_{\omega'}(t, \frac{j}{\eta}) = 0, \text{ otherwise}$$

$$\text{and let } F_{\eta} = \frac{1}{2} Id_{\overline{\mathcal{T}_{\eta}} \times \overline{\mathcal{R}_{\eta}}}$$

**Lemma 0.13.** Let  $f$  satisfy the hypotheses of Theorem 0.9, with the extra requirement on the boundary condition  $g$ , then, for finite  $t$ ;

$$\check{F}_{\omega'} * f = (hF_{\omega'}) \check{*} g$$

where  $h$  is given by;

$$h(t, x) = (1 + \frac{1}{\eta} \psi_{\eta}(x)^2)^{[nt]}$$

*Proof.* By Theorem 0.9, for finite  $t$ ,  $\hat{f} = h\hat{g}$ , and so,  $\hat{f}F_{\omega'} = hF_{\omega'}\hat{g}$ . Let  $a = (hF_{\omega'}) \check{*}$  and  $b = \check{F}_{\omega'}$ , then, by Theorem 0.11 and Remark 0.7,  $a \hat{*} g = \hat{a}\hat{g} = 2hF_{\omega'}\hat{g} = 2F_{\omega'}\hat{f}$ , and, similarly,  $b \hat{*} f = \hat{b}\hat{f} = 2F_{\omega'}\hat{f}$ . Therefore,  $a \hat{*} g = b \hat{*} f$ , and, again using Remark 0.7, we obtain  $b * f = a * g$ , as required. □

**Definition 0.14.** We call  $\Psi_{\omega'}(t, x, y) = (hF_{\omega'}) \check{*}(t, x - y)$  a nonstandard heat kernel on  $\overline{\mathcal{T}_{\eta}} \times \overline{\mathcal{R}_{\eta}}^2$ .

**Lemma 0.15.** *Let  $\Psi(t, x, y)$  be as in Definition 0.14. Then, for finite  $(t, x, y) \in \overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta^{-2}$ , with  ${}^\circ t \neq 0$ , and  $\omega' \leq \log(\eta)^{\frac{1}{2}}$ ;*

$${}^\circ \Psi(t, x, y) = \frac{1}{\sqrt{4\pi} {}^\circ t} \exp\left(\frac{-({}^\circ x - {}^\circ y)^2}{4 {}^\circ t}\right)$$

*Proof.* We first claim that, for finite  $x \in \overline{\mathcal{R}}_\eta$ ,  ${}^\circ \gamma_\eta(x) = \exp(-\pi^2 {}^\circ x^2)$ , where  $\gamma_\eta(x) = (1 + \frac{1}{\eta} \psi_\eta(x)^2)^\eta$ ,  $(*)$ . For  $y \in \mathcal{R}$ , let  $(s_n)_{n \in \mathcal{N}}$  be the standard sequence, defined by;

$$s_n(y) = \frac{\exp(\pi i \frac{y}{n}) - 1}{\frac{1}{n}} = y \left( \frac{\exp(\pi i \frac{y}{n}) - 1}{\frac{y}{n}} \right), (y \neq 0)$$

Then, for  $y \neq 0$ ;

$$\lim_{n \rightarrow \infty} (s_n(y)) = \lim_{h \rightarrow 0} y \left( \frac{\exp(\pi i h) - 1}{h} \right) = y \frac{d}{ds} \Big|_{s=0} \exp(\pi i s) = \pi i y$$

and the sequence converges uniformly in  $y$ , on bounded intervals, <sup>(1)</sup>. Now, it is standard result, <sup>(2)</sup>, that the sequence of functions  $(r_n(w))_{n \in \mathcal{N}}$ , defined by;

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<sup>1</sup> We estimate the rate of convergence of the sequence  $p_n = n(\exp(\pi i \frac{1}{n}) - 1) - i\pi$ . We have;

$$\begin{aligned} p_n &= \sum_{m \geq 2} \frac{(\pi i)^m (\frac{1}{n})^{m-1}}{m!} = \frac{-\pi^2}{n} \sum_{m \geq 0} \frac{(\frac{\pi i}{n})^m}{(m+2)!} \\ |p_n| &\leq \frac{\pi^2}{n} \sum_{m \geq 0} \frac{(\frac{\pi}{n})^m}{m!} = \frac{\pi^2}{n} \exp\left(\frac{\pi}{n}\right) \leq \frac{\pi^2 \exp(\pi)}{n} \end{aligned}$$

In particular,  $|p_n| < \epsilon$ , and, therefore,  $|s_n(y) - i\pi y| < \epsilon|y|$ , if  $n \geq \frac{\pi^2 \exp(\pi)}{\epsilon}$ . Hence,  $|s_n(y) - i\pi y| < \epsilon$ , if  $n \geq \frac{\pi^2 \exp(\pi)|y|}{\epsilon}$

<sup>2</sup> We estimate the rate of convergence of the sequence  $q_n(w) = r_n(w) - \exp(w)$ , for  $w \in \mathcal{C}$ . We have, taking a branch of the logarithm with  $\log(1) = 0$ , and cutting the complex plane from  $-\infty$  to  $-1$ , for  $n > |w|$ ;

$$\begin{aligned} \log(r_n(w)) - w &= n \log\left(1 + \frac{w}{n}\right) - w \\ |\log(r_n(w)) - w| &\leq \sum_{m=1}^{\infty} \frac{|w|^{m+1}}{(m+1)n^m} \leq \frac{|w|^2}{n} \sum_{m=0}^{\infty} \frac{|w|^m}{n^m} = \frac{|w|^2}{n} \frac{1}{1 - \frac{|w|}{n}} = \frac{|w|^2}{n - |w|} \end{aligned}$$

Moreover, observe that, for  $w \in \mathcal{C}$ ;

$$|\exp(w) - 1| \leq \sum_{m=1}^{\infty} \frac{|w|^m}{m!} = |w| \sum_{m=0}^{\infty} \frac{|w|^m}{(m+1)!} \leq |w| \exp(|w|)$$

Therefore, for  $\epsilon > 0$ ,  $|\exp(w) - 1| < \epsilon$ , if  $|w| < \min(\frac{\epsilon}{e}, 1)$ , and, for  $w', w \in \mathcal{C}$ , with  $\operatorname{Re}(w) \leq 0$ ,  $|\exp(w') - \exp(w)| < \epsilon$  if  $|\exp(w' - w) - 1| < \epsilon \leq \epsilon |\exp(-w)|$ . Hence,  $|\exp(w') - \exp(w)| < \epsilon$ , if  $|w' - w| < \min(\frac{\epsilon}{e}, 1)$ , for  $\operatorname{Re}(w) \leq 0$ .

$$r_n(w) = (1 + \frac{w}{n})^n$$

converges uniformly to  $\exp(w)$  on bounded subsets of  $\mathcal{C}$ . Therefore, if  $(t_n)_{n \in \mathcal{N}}$  is the sequence defined by  $t_n = r_n(s_n^2)$ , then;

$$\lim_{n \rightarrow \infty} t_n = \exp(-\pi^2 y^2)$$

It follows that the sequence of functions  $t_n(y)$  converges uniformly to  $\exp(-\pi^2 y^2)$  on bounded intervals of  $\mathcal{R}$ , <sup>(3)</sup> In particular, given  $N, \epsilon > 0$  standard the statement;

$$\forall y \leq N \exists M \forall n \geq M (|t_n(y) - \exp(-\pi^2 y^2)| < \epsilon)$$

is true in  $\mathcal{R}$ , therefore, by transfer, is true in  ${}^*\mathcal{R}$ . As  $\epsilon$  and  $N$  were arbitrary, it follows that, for all finite  $x \in \overline{\mathcal{R}_\eta}$ ;

$$\gamma_\eta(x) \simeq t_\eta(x) \simeq {}^*\exp(-\pi^2 x^2) \simeq \exp(-\pi^{2\circ} x^2)$$

using continuity of  $\exp$ , and Theorem 2.25 of [7] or [5]. Therefore,  $(*)$  holds. Now, by continuity of the function  $q(w) = w^s$ , for  $s \in \mathcal{R}$ , and the fact that  $\frac{\eta^\circ t - [\eta t]}{\eta} \simeq 0$ , for finite  $t \in \overline{\mathcal{R}_\eta}$ , it follows, again using [5] or Theorem 2.25 of [7], that, for finite  $(t, x) \in \overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta}$ ,  ${}^\circ h(t, x) = \exp(-\pi^{2\circ} t^\circ x^2)$ ,  $(**)$ .

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So, for  $Re(w) \leq 0$ , if  $\frac{|w|^2}{n-|w|} < \min(\frac{\epsilon}{e}, 1)$ , that is  $n > |w| + |w|^2 \max(1, \frac{\epsilon}{e})$ , then  $|q_n(w)| = |r_n(w) - \exp(w)| < \epsilon$ .

<sup>3</sup> We estimate the rate of convergence of the sequence  $b_n(y) = t_n(y) - \exp(-\pi^2 y^2)$ , for  $y \in \mathcal{R}$ ,  $y \neq 0$ . It is a straightforward calculation, to show that, if  $|s_n(y) - i\pi y| < \min(2|y|, \frac{\epsilon}{3|y|})$ , then  $|s_n^2(y) - (-\pi^2 y^2)| < \epsilon$ . Combining this with the result of footnote 1, we obtain that if  $n > \max(\frac{\pi^2 \exp(\pi)}{2}, \frac{3\pi^2 \exp(\pi) y^2}{\epsilon})$ ,  $(*)$ , then  $|s_n^2(y) - (-\pi^2 y^2)| < \epsilon$ . Using footnote 2, we also have that if  $\epsilon < \min(\frac{\delta}{e}, 1)$ , then  $|\exp(s_n^2(y)) - \exp(-\pi^2 y^2)| < \delta$ ,  $(**)$ . Now, assuming  $(*)$  is satisfied, we have  $|s_n(y)| < (\epsilon + \pi^2 y^2)^{\frac{1}{2}}$ . Then, using footnote 2, if  $\epsilon < \min(\frac{\delta}{e}, 1, \frac{\pi^2 y^2}{2})$ ,  $(\dagger)$ ,  $n > \max((\epsilon + \pi^2 y^2)^{\frac{1}{2}} + (\epsilon + \pi^2 y^2) \max(1, \frac{\epsilon}{\delta}), \max(\frac{\pi^2 \exp(\pi)}{2}, \frac{3\pi^2 \exp(\pi) y^2}{\epsilon}))$ ,  $(\dagger\dagger)$ , then  $|\exp(s_n^2(y)) - \exp(-\pi^2 y^2)| < \delta$ ,  $|r_n(s_n^2(y)) - \exp(s_n^2(y))| < \delta$ , so  $|b_n(y)| = |t_n(y) - \exp(-\pi^2 y^2)| = |r_n(s_n^2(y)) - \exp(-\pi^2 y^2)| < 2\delta$ ,  $(***)$ . Now, if  $\delta < \min(e, \pi^2 y^2 e)$ , we can satisfy  $(\dagger)$  by taking  $\epsilon = \frac{\delta}{2e}$ . Substituting into  $(\dagger\dagger)$ , we obtain, if  $n > \max((1 + \pi^2 y^2)^{\frac{1}{2}} + (1 + \pi^2 y^2) \frac{\epsilon}{\delta}, \frac{\pi^2 \exp(\pi)}{2}, \frac{6\pi^2 \exp(\pi) y^2 e}{\delta})$ , then  $(***)$  holds. Taking  $\delta = \frac{1}{2|y|^r}$ , for  $r \in \mathcal{N}$ , there exist constants  $C_2, C_3 > 0$ , such that, for all  $y \in \mathcal{R}$ , if  $n > \max(C_2, C_3 |y|^{r+2})$ , then  $|b_n(y)| < \frac{1}{|y|^r}$ .

Now if  $\omega'' \in {}^*\mathcal{R} \setminus \mathcal{R}$ , with  $|\omega''| \leq \eta^{\frac{1}{4}}$ , then, in particular,  $\eta > \max(C_2, C_3|\omega''|^3)$ , see footnote 3. Hence, we have, by transfer, that;

$$|t_\eta(\omega'') - {}^*\exp(-\pi^2\omega''^2)| < \frac{1}{|\omega''|} \simeq 0$$

for infinite  $\omega''$ . As  $\lim_{x \rightarrow \infty} \exp(-\pi^2 x^2) = 0$ , it is a standard result, see [5], that  ${}^*\exp(-\pi^2\omega''^2) \simeq 0$ , hence  $|t_\eta(\omega'')| \simeq 0$ , and  ${}^\circ t_\eta(\omega'') = 0$ . Now, by a similar argument to the above, for finite  $t$ , with  ${}^\circ t \neq 0$ , we have  $h(t, \omega'') \simeq 0$ . Combining these results, we have that;

$${}^\circ hF_{\omega'}|_{st^{-1}(\mathcal{T}_{>0}) \times \overline{\mathcal{R}}_\eta} = st^*(\exp(-\pi^2 tx^2)_\infty) \ (\sharp)$$

for  $\omega' \in {}^*\mathcal{N} \setminus \mathcal{N}$ , with  $\omega' \leq \eta^{\frac{1}{4}}$ . Here, we adopt the notation in Definition 0.5 of [6], letting  $\exp(-\pi^2 tx^2)_\infty$  denote the extension of  $\exp(-\pi^2 tx^2)$  on  $\mathcal{T}_{>0} \times \mathcal{R}$  to  $\mathcal{T}_{>0}^{+\infty} \times \mathcal{R}^{+-\infty}$ , by setting  $\exp(-\pi^2 tx^2)_\infty = 0$ , at infinite values.

Now, for finite  $(t, x) \in \overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta$ , we have, by (\*\*), that;

$$|h(t, x)| \leq 2 {}^*\exp(-\pi^2 tx^2) \ (\dagger)$$

For  $(t, x) \in \overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta$ , with  $t > 1$  finite, and  $x$  infinite, with  $x \leq \eta^{\frac{1}{5}}$ , we have, using footnote 3, that;

$$|t_\eta(x)|^t < |t_\eta(x)| \leq |t_\eta(x)| \leq {}^*\exp(-\pi^2 x^2) + \frac{1}{x^2} \leq \frac{C}{x^2} \ (\dagger\dagger)$$

where  $C \in \mathcal{R}_{>0}$ .

For  $(t, x) \in \overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta$ , with  $\frac{1}{r} < {}^\circ t \leq 1$ ,  $r \in \mathcal{N}$  and  $x$  infinite, with  $x \leq \eta^{\frac{1}{2r+3}}$ , we have, by footnote 3, that;

$$|t_\eta(x)| \leq {}^*\exp(-\pi^2 x^2) + \frac{1}{x^{2r}} \leq \frac{C'}{x^{2r}}$$

$$|t_\eta(x)|^t < |t_\eta(x)|^{\frac{1}{r}} \leq \frac{C''}{x^2} \ (\dagger\dagger\dagger)$$

where  $C', C'' \in \mathcal{R}_{>0}$ . Combining the estimates, ( $\dagger$ ), ( $\dagger\dagger$ ), ( $\dagger\dagger\dagger$ ), and, using the fact that  $h(x, t)$  is the measurable counterpart of  $t_\eta(x)^t$ , we have, for  $\omega' \leq (\log(\eta))^{\frac{1}{2}}$ , and  $t$  finite,  $0 < {}^\circ t$ , that;

$$|(hF_{\omega'})_t| \leq f_{t, \eta}$$

Here,  $f_{t,\eta} : \overline{\mathcal{R}}_\eta \rightarrow \overline{\mathcal{R}}_\eta$  is the measurable counterpart of the  $*$ -continuous function  $f_t : {}^*\mathcal{R} \rightarrow {}^*\mathcal{R}$  given by;

$$f_t(x) = C_t \text{ if } |x| \leq 1$$

$$f_t(x) = \frac{C_t}{x^2} \text{ if } |x| > 1$$

and  $C_t \in \mathcal{R}_{>2}$ , depends on  $t$ . Now, using the proof of Theorem 0.17 in [6], it follows that  $f_{t,\eta}$  is  $S$ -integrable. Then, using [1], Corollary 5, it follows that  $(hF_{\omega'})_t(w)$  and  $(hF_{\omega'})_t(w) \exp_\eta(\pi i w z)$  are  $S$ -integrable,  $d\lambda_\eta(w)$ , for finite  $z \in \overline{\mathcal{R}}_\eta$ . Moreover, using [7], Theorem 3.24, and  $(\sharp)$ , we have, for finite  $(t, z) \in \overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta$ , with  ${}^\circ t \neq 0$ , that;

$$\begin{aligned} {}^\circ(hF_{\omega'})(t, z) &= {}^\circ \int_{\overline{\mathcal{R}}_\eta} hF_{\omega'}(t, w) \exp_\eta(\pi i w z) d\lambda_\eta(w) \\ &= \frac{1}{2} \int_{w \text{ finite}} {}^\circ h \exp_\eta(\pi i w ({}^\circ z)) dL(\lambda_\eta)(w) \\ &= \frac{1}{2} \int_{\mathcal{R}} \exp(-\pi^2 {}^\circ t {}^\circ w^2) \exp(\pi i w ({}^\circ z)) d\mu(w) \ (\sharp\sharp) \\ &= \frac{1}{\sqrt{4\pi {}^\circ t}} \exp\left(\frac{-({}^\circ z)^2}{4 {}^\circ t}\right), \ (4). \end{aligned}$$

Now substituting  $x - y$  for  $z$ , we obtain the result. □

**Definition 0.16.** Let  $g : \mathcal{R} \rightarrow \mathcal{C}$  be a continuous function, satisfying the growth condition;

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<sup>4</sup> Taking standard parts, the fact that;

$$\int_{\mathcal{R}} e^{i\pi w z - \pi^2 t w^2} dw = \frac{1}{\sqrt{\pi t}} e^{\frac{-z^2}{4t}}$$

is a standard result, which we include for want of a convenient reference. We have  $i\pi w z - \pi^2 t w^2 = -\pi^2 t (w - \frac{iz}{2\pi t})^2 - \frac{z^2}{4t}$ . Hence;

$$\begin{aligned} &\int_{\mathcal{R}} e^{i\pi w z - \pi^2 t w^2} dw \\ &= e^{\frac{-z^2}{4t}} \int_{\mathcal{R}} e^{-\pi^2 t (w - \frac{iz}{2\pi t})^2} dw \\ &= e^{\frac{-z^2}{4t}} \int_{Im(w') = \frac{-z}{2\pi t}} e^{-\pi^2 t w'^2} dw' \ (w' = w - \frac{iz}{2\pi t}) \\ &= \frac{e^{\frac{-z^2}{4t}}}{\pi \sqrt{t}} \int_{Im(w'') = \frac{-\pi \sqrt{t} z}{2\pi t}} e^{-w''^2} dw'' \ (w'' = \pi \sqrt{t} w') \\ &= \frac{1}{\sqrt{\pi t}} e^{\frac{-z^2}{4t}} \end{aligned}$$

$$|g(x)| \leq A \exp(B|x|^\rho), \quad (x \in \mathcal{R})$$

for some constants  $A, B$  and  $\rho < 2$ . Then the function  $H : \mathcal{T} \times \mathcal{R} \rightarrow \mathcal{C}$ , defined by;

$$H(0, x) = g(x)$$

$$H(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathcal{R}} \exp\left(\frac{-(x-y)^2}{4t}\right) g(y) d\mu(y) \quad (t > 0)$$

which is continuous, and satisfies the standard heat equation;

$$\frac{\partial H}{\partial t} - \frac{\partial^2 H}{\partial x^2} = 0$$

on  $\mathcal{T}_{>0} \times \mathcal{R}$ , <sup>(5)</sup> is known as the classical solution to the heat equation with boundary condition  $g$ .

**Theorem 0.17.** Let  $g$  be as in Definition 0.16, let  $g_\eta$  denote its measurable extension to  $\overline{\mathcal{R}_\eta}$ , and, let  $g_{\eta,\omega}$  be the truncation of  $g_\eta$ , given by;

$$g_{\eta,\omega} = g_\eta \chi_{[-\omega,\omega]}$$

for a nonstandard step function  $\chi_{[-\omega,\omega]}$ , with  $\omega \in {}^*\mathcal{N} \setminus \mathcal{N}$ ,  $\eta - \omega$  infinite and  $\omega < \omega^{\frac{1}{2}}$ . Then, with  $f$  determined by Lemma 0.5, for  $g_{\eta,\omega}$  as the boundary condition, we have;

$${}^\circ(F_{\omega'} * f)|_{st^{-1}(\mathcal{T}_{>0} \times \mathcal{R})} = st^*(H_\infty)$$

if  $\omega' < \log(\eta)^{\frac{1}{2}}$ , and  $H_\infty$  is obtained from the classical solution  $H$  of the heat equation, with boundary condition  $g$ , given in Definition 0.16.

*Proof.* Using the following footnote 6, we obtain, by transfer and the measurability observation at the end of Lemma 0.15, that, for any given  $\delta', t \in {}^*\mathcal{R}_{>0}, x \in {}^*\mathcal{R}$ ;

$$|h(t, x) - \exp_\eta(-\pi^2 t x^2)| < \delta', \quad (*)$$

if  $\eta > C_4(x, \delta', t)$ . In particular, observing that the function  $C_4 : {}^*\mathcal{R} \times {}^*\mathcal{R}_{>0}^2$  is increasing in  $x$  and  $t$ , we have, for a given infinite  $\omega' \in {}^*\mathcal{N}$ ,

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<sup>5</sup> A good proof of this fact can be found in [8], (Theorem 2.1), if  $g \in S(\mathcal{R})$ . For the more general case, see [3]

that (\*) holds for all  $|x| \leq \omega'$ , and finite  $t$ , if  $\eta > C_4(\omega', \delta', \omega'), (6)$ .

In particular, we obtain that;

$$\begin{aligned} |(hF_{\omega'})(t, z) - \check{\theta}(t, z)| &\leq \int_{\overline{\mathcal{R}}_\eta} |hF_{\omega'}(t, w) - \theta(t, w)| d\lambda(w) \\ &\leq 2 \cdot \frac{1}{2} \delta' \omega' = \delta' \omega' \quad (**) \end{aligned}$$

for all  $z \in \overline{\mathcal{R}}_\eta$  and finite  $t \in \overline{\mathcal{T}}_\eta$ ,  $t \neq 0$ , where  $\theta(t, w) = \exp_\eta(-\pi^2 t w^2) F_{\omega'}(t, w)$ . Now, substituting  $x - y$  for  $z$  in (\*\*), and multiplying through by

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<sup>6</sup> Taking a principal branch of the logarithm, we have, for  $w, w' \in \mathcal{C}$ , with  $w \neq 0$  and  $|w' - w| < \frac{|w|}{2}$ , that the function  $\theta(t) = \log(w + t(w' - w))$  is continuously differentiable on the interval  $[0, 1]$ , with;

$$\theta'(t) = \frac{w' - w}{w + t(w' - w)}$$

Applying the mean value theorem to the real and imaginary parts of  $f$ , we obtain;

$$|\log(w) - \log(w')| = |\theta(1) - \theta(0)| \leq 2 \frac{|w - w'|}{\min_{t \in [0, 1]} |w + t(w' - w)|} \leq 4 \frac{|w - w'|}{|w|} \quad (*)$$

Using footnote 2, we have, for  $w, w' \in \mathcal{C}$ , with  $|w - w'| < 1$  and  $\operatorname{Re}(w) \leq 0$ , that;

$$|\exp(w) - \exp(w')| = |\exp(w)| |\exp(w' - w) - 1| \leq |w' - w| \exp(|w' - w|) |\exp(w)| \leq e |w - w'| \quad (**)$$

Now, for  $t \in \mathcal{R}$ ,  $t > 0$ , we can satisfy the condition  $|t \log(w') - t \log(w)| < 1$ , using (\*) and assuming that  $|w' - w| < \frac{|w|}{2}$ , by taking  $|w' - w| < \frac{|w|}{4t}$ . Then, assuming, that  $|w| \leq 1$ , so that  $\operatorname{Re}(t \log(w)) \leq 0$ , and  $w \neq 0$ , we have, combining (\*), (\*\*), that;

$$|w'^t - w^t| = |\exp(t \log(w')) - \exp(t \log(w))| < 4et \frac{|w' - w|}{|w|}$$

if  $|w' - w| < \min(\frac{|w|}{2}, \frac{|w|}{4t})$ , ( $\dagger$ ). We now estimate the rate of convergence of the sequence  $v_n(y) = t_n(y)^t - \exp(-\pi^2 t y^2)$ , for  $t \in \mathcal{R}_{>0}$ . Let  $C(\delta, y)$  be the constant obtained in footnote 3, so that there (\*\*\*) holds. Then, it is easy to see, using ( $\dagger$ ) and the fact that  $0 < \exp(-\pi^2 y^2) \leq 1$ , that, if,  $n > \max(C(\frac{\exp(-\pi^2 y^2)}{4}, y), C(\frac{\exp(-\pi^2 y^2)}{8t}, y), C(\delta' \frac{\exp(-\pi^2 y^2)}{8et}, y))$ , then;

$$|v_n(y)| = |t_n(y)^t - \exp(-\pi^2 t y^2)| < \delta' \quad (\dagger\dagger)$$

In particular, substituting into the expression for  $C(\delta, y)$ , we can find constants  $C_2, C_3 \in \mathcal{R}$ , such that ( $\dagger\dagger$ ) holds, for;

$$n > \max(C_2, \frac{C_3 y^2 \exp(\pi^2 y^2) t}{\delta'}) = C_4(y, \delta', t)$$

$g_{\eta,\omega}(y)$ , we have, from (\*\*), that;

$$|(hF_{\omega'})^\sim(t, x - y)g_{\eta,\omega}(y)| \leq |\check{\theta}(t, x - y)g_{\eta,\omega}(y)| + \delta'\omega'|g_{\eta,\omega}(y)| \quad (***)$$

for all  $x, y \in \overline{\mathcal{R}}_\eta$  and  $t$  as above. Now using the growth condition in Definition 0.16, we have that  $|\delta'\omega'g_{\eta,\omega}| \leq \frac{\chi_{[-\omega,\omega]}}{\omega^2}$ , if  $\delta' \leq \frac{*exp(-B|\omega|^\rho)}{A\omega^2\omega'}$ . Using [7](Theorem 3.24), the fact that  $\int_{\overline{\mathcal{R}}_\eta} \frac{\chi_{[-\omega,\omega]}}{\omega^2} d\lambda = \frac{2}{\omega} \simeq 0$ , and [1], we have  $\delta'\omega'|g_{\eta,\omega}|$  is  $S$ -integrable, and  $\int_{\overline{\mathcal{R}}_\eta} \delta'\omega'|g_{\eta,\omega}| d\lambda \simeq 0$ . In particular, using Definition 0.10 and Lemma 0.13, this implies that;

$$(\check{F}_{\omega'} * f)(t, x) = (hF_{\omega'})^\sim * g_{\eta,\omega}(t, x) \simeq \check{\theta} * g_{\eta,\omega}(t, x) \quad (****)$$

for all finite  $t \in \overline{\mathcal{R}}_{\eta,>0}$  and  $x \in \mathcal{R}_\eta$ . Let  $\tau(t, w) = exp_\eta(-\pi^2 tw^2)$ , then, using the following footnote 7, we obtain by transfer;

$$|\check{\tau}(t, z) - \frac{1}{\sqrt{\pi t}} exp_\eta(\frac{-z^2}{4t})| \leq \frac{K(t)}{\eta} + G(t) \frac{|z|}{\eta} + H \frac{z^2}{\eta}$$



for  $t \in \overline{\mathcal{R}}_{\eta, > 0}$  and  $z \in \overline{\mathcal{R}}_{\eta, (7)}$ . In particular, if  $\omega'' \in {}^*\mathcal{N}$  is infinite,  $\delta'' \in {}^*\mathcal{R}_{> 0}$ , then we obtain;

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<sup>7</sup> We require the following estimate, see [6] for relevant terminology. Let  $f : \mathcal{R} \rightarrow \mathcal{R}$  be differentiable on  $\mathcal{R}$  and increasing (decreasing)  $(*)$  on the interval  $[\frac{i}{n}, \frac{j}{n}]$ , where  $i, j \in \mathcal{Z}$ ,  $-n^2 \leq i < j \leq n^2$ , and  $n \in \mathcal{N}$ , then;

$$\begin{aligned}
& \left| \int_{[\frac{i}{n}, \frac{j}{n}]} f_n d\lambda_n - \int_{[\frac{i}{n}, \frac{j}{n}]} f d\mu \right| \\
& \leq \frac{1}{n} \sum_{k=0}^{j-i-1} \left| f\left(\frac{i+k+1}{n}\right) - f\left(\frac{i+k}{n}\right) \right| (**) \\
& = \frac{1}{n} \sum_{k=0}^{j-i-1} \left| \int_{\frac{i+k}{n}}^{\frac{i+k+1}{n}} f' d\mu \right| (***) \\
& \leq \frac{1}{n} \sum_{k=0}^{j-i-1} \int_{\frac{i+k}{n}}^{\frac{i+k+1}{n}} |f'| d\mu \\
& = \frac{1}{n} \int_{\frac{i}{n}}^{\frac{j}{n}} |f'| d\mu (****)
\end{aligned}$$

where, in  $(**)$ , we have used the assumption  $(*)$  and the definition of the relevant integrals, and, in  $(***)$ , we have used the Fundamental Theorem of Calculus. Now let  $Y(x) = \exp(-\pi^2 t x^2) \cos(\pi x z)$  and let  $Y_n(x)$  be its  $\lambda_n(x)$  measurable counterpart on  $\mathcal{R}_n$ , where  $t \in \mathcal{R}_{> 0}$  and  $z = \frac{j}{n}$ ,  $0 < j \leq n^2 - 1$ . Observe that the zeros of  $Y$  on  $[0, n]$  are located at the points  $p_k = \frac{(2k-1)n}{2j}$ , for  $k \in \mathcal{N} \cap [0, j]$ , and, the local maxima (minima) of  $Y$  on  $[0, n]$ , are located at points  $q_k$ , where  $p_k < q_k < p_{k+1}$ , for  $0 \leq k \leq j-1$ , and  $p_j < q_j < n$ , for  $n \geq D$ , some  $D \in \mathcal{R}$ . Let  $p'_k$  denote the points  $\frac{[np_k]}{n}$ ,  $p''_k$  the points  $p'_k + \frac{1}{n}$ , and, similarly, define  $q'_k, q''_k$ , then, it is easy to see (check this) that we can choose a constant  $D(t)$ , such that  $0 < p'_k < p''_k < q'_k < q''_k < p'_{k+1} < n$ , for  $0 \leq k \leq j-1$ , and  $p'_j < q'_j < q''_j < n$ , for  $n \geq \max(D(t), \sqrt{2j})$ . Now, using  $(****)$ , and the fact that  $Y$  is monotone on the intervals  $[p''_k, q'_k]$ ,  $[q''_k, p'_{k+1}]$ , for  $0 \leq k \leq j-1$ , and on  $[0, p'_0]$ ,  $[p''_j, q'_j]$ ,  $[q''_j, n]$ , we obtain;

$$\left| \int_{[p''_k, q'_k]} Y_n d\lambda_n - \int_{[p''_k, q'_k]} Y d\mu \right| \leq \frac{1}{n} \int_{[p''_k, q'_k]} |Y'| d\mu (\dagger)$$

and, similarly, for the other intervals. Choose a constant  $A(t) \in \mathcal{R}$ , such that  $|Y(x)| \leq \frac{1}{x^2}$ ,  $(\#)$ , for  $|x| > A(t)$ . Let  $k_{max}$  be the largest  $k$  such that  $p''_k \leq A(t)$ , then  $\frac{(2k_{max}-1)n}{2j} + \frac{1}{n} \leq A(t)$  and  $k_{max} \leq \frac{jC(t)}{n} + 1$ . Let  $U = \bigcup_{0 \leq k \leq k_{max}} [p'_k, p''_k] \cup [q'_k, q''_k]$ , then, using the bound  $|Y| \leq 1$ ;

$$\left| \int_U Y_n d\lambda_n \right| \leq \frac{1}{n} 2k_{max} \leq \frac{2jC(t)}{n^2} + \frac{2}{n} (\dagger\dagger)$$

and, similarly, for  $\left| \int_U Y d\mu \right|$ . Let  $V = \bigcup_{k_{max} < k \leq j} [p'_k, p''_k] \cup [q'_k, q''_k]$ , then, using the bound  $(\#)$ ;

$$\left| \int_V Y_n d\lambda_n \right| \leq \frac{2}{n} \sum_{k_{max} < k \leq j} \frac{1}{\left(\frac{(2k-1)n}{2j}\right)^2} \leq 16 \frac{j^2}{n^3} (\dagger\dagger\dagger)$$

and, similarly, for  $\left| \int_V Y d\mu \right|$ . Let  $W = (\bigcup_{0 \leq k \leq j-1} [p''_k, q'_k] \cup [q''_k, p'_{k+1}]) \cup [0, p'_0] \cup [p''_j, q'_j] \cup [q''_j, n]$ . Then, using  $(\dagger)$ , we have;

$$|\frac{1}{2}\tilde{\tau}(t, z) - \frac{1}{\sqrt{4\pi t}}\exp_{\eta}(\frac{-z^2}{4t})| < \delta'' \quad (\dagger)$$

for all finite  $t \in \overline{\mathcal{R}}_{\eta>0}$ , and  $z \in \overline{\mathcal{R}}_{\eta}$ , with  $|z| \leq \omega''$ , if  $\eta > \frac{3\omega''^3}{\delta''}$ ,  $(\dagger\dagger)$ . Using Definition 0.6, and transfer of the following footnote 8, we have;

$$\begin{aligned} |\frac{1}{2}\tilde{\tau}(t, z) - \check{\theta}(t, z)| &\leq \frac{1}{2} \int_{|x| \geq \omega'} \exp_{\eta}(-\pi^2 t x^2) d\lambda_{\eta}(x) \\ &\leq \frac{1}{2\pi\sqrt{t}} * \exp(-\pi^2 t (\omega' - \frac{1}{\eta})^2) \end{aligned}$$

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$$|\int_W Y_n d\lambda_n - \int_W Y d\mu| \leq \frac{1}{n} \int_W |Y'| d\mu \leq \frac{1}{n} \int_{[0,n)} |Y'| d\mu \quad (\dagger\dagger\dagger\dagger)$$

Using  $(\dagger\dagger)$ ,  $(\dagger\dagger\dagger)$ ,  $(\dagger\dagger\dagger\dagger)$ , and the fact that  $U, V, W$  is a partition of  $[0, n)$ , we obtain;

$$|\int_{[0,n)} Y_n d\lambda_n - \int_{[0,n)} Y d\mu| \leq \frac{1}{n} \int_{[0,n)} |Y'| d\mu + \frac{4jC(t)}{n^2} + \frac{4}{n} + 32\frac{j^2}{n^3} \quad (\#\#)$$

Then, as  $Y$  is even,  $|Y| \leq 1$ ,  $|Y'| \leq \exp(-\pi^2 t x^2)(2\pi^2 t |x| + \frac{\pi j}{n})$ , we obtain, using  $(\#\#)$ ;

$$\begin{aligned} &|\int_{\mathcal{R}_n} Y_n d\lambda_n - \int_{[-n,n]} Y d\mu| \\ &\leq 2|\int_{[0,n)} Y_n d\lambda_n - \int_{[0,n)} Y d\mu| + \frac{1}{n}(|Y(-n)| + |Y(0)|) \\ &\leq \frac{1}{n} \int_{\mathcal{R}} |Y'| d\mu + \frac{8jC(t)}{n^2} + \frac{8}{n} + 64\frac{j^2}{n^3} + \frac{2}{n} \\ &\leq \frac{D(t)\pi j}{n^2} + \frac{2E(t)\pi^2 t}{n} + \frac{8jC(t)}{n^2} + \frac{10}{n} + 64\frac{j^2}{n^3} = \frac{F(t)}{n} + G(t)\frac{j}{n^2} + H\frac{j^2}{n^3} \quad (\#\#\#) \end{aligned}$$

where  $F(t), G(t), H \in \mathcal{R}$ . Choosing a constant  $I(t) \in \mathcal{R}$  such that  $\exp(-\pi^2 t x^2) \leq \frac{I(t)}{2x^2}$ , for  $|x| > 1$ , we obtain;

$$\begin{aligned} &|\int_{\mathcal{R}_n} Y_n d\lambda_n - \int_{\mathcal{R}} Y d\mu| \\ &\leq \frac{F(t)}{n} + G(t)\frac{j}{n^2} + H\frac{j^2}{n^3} + \frac{I(t)}{n} = \frac{J(t)}{n} + G(t)\frac{j}{n^2} + H\frac{j^2}{n^3} \quad (\#\#\#\#) \end{aligned}$$

Let  $Z(x) = \exp(-\pi^2 t x^2) \sin(\pi x z)$ , with hypotheses and  $Z_n(x)$  as above. Then, as  $Z$  is odd,  $|Z| \leq 1$ , we have  $\int_{\mathcal{R}} Z d\mu = 0$ ,  $\int_{\mathcal{R}_n} Z_n d\lambda_n = \frac{Z(-n)}{n}$ , and;

$$|\int_{\mathcal{R}_n} Z_n d\lambda_n - \int_{\mathcal{R}} Z d\mu| \leq \frac{1}{n} \quad (\#\#\#\#\#)$$

Let  $X(x) = \exp(-\pi^2 t x^2) \exp(i\pi x z)$ , with hypotheses and  $X_n(x)$  as above. Then, using the estimates  $(\#\#\#\#)$ ,  $(\#\#\#\#\#)$  and footnote 4, we obtain;

$$|\int_{\mathcal{R}_n} X_n d\lambda_n - \frac{1}{\sqrt{\pi t}} e^{-\frac{z^2}{4t}}| \leq \frac{K(t)}{n} + G(t)\frac{j}{n^2} + H\frac{j^2}{n^3} \quad (\#\#\#\#\#\#)$$

where  $K(t) = J(t) + 1$  and  $z = \frac{j}{n}$ .

$$\leq \omega'^* \exp(-\pi^2 t (\omega' - 1)^2) \ (\dagger\dagger\dagger)$$

for all  $z \in \overline{\mathcal{R}_\eta}$ , finite  $t \in \overline{\mathcal{R}_{\eta>0}}$ ,  $\omega' \in {}^*\mathcal{N}$  infinite,<sup>(8)</sup>.

Combining  $(\dagger)$  and  $(\dagger\dagger\dagger)$ , gives;

$$|\check{\theta}(t, z) - \Gamma(t, z)| \leq \delta'' + \omega'^* \exp(-\pi^2 t (\omega' - 1)^2) \ (\dagger\dagger\dagger\dagger)$$

for all finite  $t \in \overline{\mathcal{R}_{\eta>0}}$ , and  $|z| \leq \omega''$ ,  $z \in \overline{\mathcal{R}_\eta}$ , if the condition  $(\dagger\dagger)$  holds, where  $\Gamma(t, z) = \frac{1}{\sqrt{4\pi t}} \exp_\eta(\frac{-z^2}{4t})$ . We have, using Definition 0.10 and  $(\dagger\dagger\dagger\dagger)$ ;

$$\begin{aligned} & |(\check{\theta} * g_{\eta, \omega})(t, x) - (\Gamma * g_{\eta, \omega})(t, x)| \\ &= |\int_{\overline{\mathcal{R}_\eta}} (\check{\theta} - \Gamma)(x - y) g_{\eta, \omega}(y) d\lambda_\eta(y)| \\ &\leq \int_{\overline{\mathcal{R}_\eta}} (\delta'' + \omega'^* \exp(-\pi^2 t (\omega' - 1)^2)) |g_{\eta, \omega}(y)| d\lambda_\eta(y) \ (\#) \end{aligned}$$

for finite  $t \in \overline{\mathcal{R}_{\eta>0}}$ , finite  $x \in \overline{\mathcal{R}_\eta}$ , if  $\omega'' = 2\omega$ , that is, from  $(\dagger\dagger)$ ,  $\eta > \frac{24\omega^3}{\delta''}$ ,  $(\#\#)$ . Following the same argument as above, we have  $\delta'' g_{\eta, \omega}$  is  $S$ -integrable and  $|\delta'' g_{\eta, \omega}| \leq \frac{\chi_{[-\omega, \omega]}}{\omega^2}$ , if  $\delta'' \leq \frac{{}^*\exp(-B\omega^\rho)}{\omega^2}$ , so we require, from  $(\#\#)$ , that  $\eta > 24\omega^{5*} \exp(B\omega^\rho)$ ,  $(\#\#\#)$ . Similarly, we have  $\omega'^* \exp(-\pi^2 t (\omega' - 1)^2) g_{\eta, \omega}$  is  $S$ -integrable and  $|\omega'^* \exp(-\pi^2 t (\omega' - 1)^2) g_{\eta, \omega}| \leq \frac{\chi_{[-\omega, \omega]}}{\omega^2}$ , if  ${}^*\exp(-\pi^2 t (\omega' - 1)^2 + 1) \leq \frac{{}^*\exp(-B\omega^\rho)}{\omega^2}$ . By a simple calculation, this can be achieved if  $\omega' \geq C \max(\log(\omega)^{\frac{1}{2}}, \omega^{\frac{\rho+1}{2}})$ ,

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<sup>8</sup> We make the following estimate, with  $t \in \mathcal{R}_{>0}$ ;

$$\begin{aligned} & \int_{|x| \geq \frac{j}{n}, x \in \mathcal{R}_n} \exp_n(-\pi^2 t x^2) d\lambda_n(x) \\ &\leq \frac{2}{n} \sum_{k=j}^{n^2} \exp(-\pi^2 t (\frac{k}{n})^2) \\ &\leq 2 \int_{\frac{j-1}{n}}^n \exp(-\pi^2 t x^2) d\mu(x) \\ &\leq 2 \int_{\frac{j-1}{n}}^\infty \exp(-\pi^2 t x^2) d\mu(x) \\ &= 2 \int_{\pi^2 t (\frac{j-1}{n})^2}^\infty \frac{\exp(-u)}{2\pi\sqrt{tu}} d\mu(u), \ (u = \pi^2 t x^2) \\ &\leq \frac{1}{\pi\sqrt{t}} \int_{\pi^2 t (\frac{j-1}{n})^2}^\infty \exp(-u) d\mu(u), \ (\frac{j-1}{n} \geq \frac{1}{\pi\sqrt{t}}) \\ &\leq \frac{1}{\pi\sqrt{t}} \exp(-\pi^2 t (\frac{j-1}{n})^2) \end{aligned}$$

(####). If both the conditions (###) and (####) are satisfied, we then have;

$$(\check{\theta} * g_{\eta,\omega})(t, x) \simeq (\Gamma * g_{\eta,\omega})(t, x) \text{ (#####)}$$

for finite  $t \in \overline{\mathcal{R}_{\eta>0}}$ , finite  $x \in \overline{\mathcal{R}_{\eta}}$ . Finally, using Definition 0.10, we have;

$$\Gamma * g_{\eta,\omega}(t, x) = \int_{\overline{\mathcal{R}_{\eta}}} \Gamma(t, x - y) g_{\eta,\omega}(y) d\lambda_{\eta}(y) \text{ (!)}$$

By the growth condition on  $g$ , for  $x \in \mathcal{R}$ ,  $t \in \mathcal{R}_{>0}$ , if  $\Psi(t, x - y)$  denotes the standard heat kernel, the function  $\Psi(t, x - y)g(y) : \mathcal{R} \rightarrow \mathcal{C}$  is continuous and satisfies the tail estimate  $|\Psi(t, x - y)g(y)| \leq \frac{1}{y^2}$  for sufficiently large  $|y| \geq A(t)$ ,  $A(t) \in \mathcal{R}$ . Using the proof of Theorem 0.17 in [6] and Theorem 3.24 of [7], we obtain that  $\Gamma(t, x - y)g_{\eta,\omega}(y)$  is  $S$ -integrable and  ${}^{\circ}(\Gamma * g_{\eta,\omega})(t, x) = H(t, x)$ , (!! ). For finite  $x \in \overline{\mathcal{R}_{\eta}}$ , finite  $t \in \overline{\mathcal{R}_{\eta>0}}$ , and  ${}^{\circ}t > 0$ , we have that  $\Gamma(t, x - y)g_{\eta,\omega}(y)$  is  $S$ -integrable, (!!! ). In order to see this, choose  $0 < t_1 < t < t_2$ , with  $t_1, t_2 \in \mathcal{R}$ , and  $x_1 < x < x_2$ , with  $x_1, x_2 \in \mathcal{R}$ . We then have;

$$\Gamma(t, x - y)|g_{\eta,\omega}(y)| \leq \sqrt{\frac{t_1}{t_2}} \Gamma(t_2, x_1 - y)|g_{\eta,\omega}(y)|, \text{ for } y \leq x_1$$

$$\Gamma(t, x - y)|g_{\eta,\omega}(y)| \leq \sqrt{\frac{t_2}{t_1}} \Gamma(t_2, x_2 - y)|g_{\eta,\omega}(y)|, \text{ for } y \geq x_2$$

$$\Gamma(t, x - y)|g_{\eta,\omega}(y)| \leq C(t), \text{ for } x_1 \leq y \leq x_2 \text{ (!!!!)}$$

where  $C(t) \in \mathcal{R}$ , and in (!!!!), we have used the fact that  $g$  is continuous. Now applying the result of (!! ) and using [1] (Corollary 5), we obtain (!!! ). Then, again using Theorem 3.24 of [7], we have that  ${}^{\circ}(\Gamma * g_{\eta,\omega})(t, x) = H({}^{\circ}t, {}^{\circ}x)$ , (!!!! ). Combining (\*\*\*), (####) and (!!!! ), we obtain that;

$${}^{\circ}(\check{F}_{\omega} * f)(t, x) = H({}^{\circ}t, {}^{\circ}x) \text{ (A)}$$

for finite  $x \in \overline{\mathcal{R}_{\eta}}$ , finite  $t \in \overline{\mathcal{R}_{\eta>0}}$ , under the conditions;

$$\eta > \max(C_4(\omega', \frac{*exp(-B|\omega|^{\rho})}{\omega^2\omega'}, \omega'), 25\omega^{5*}exp(B|\omega|^{\rho})) \text{ (B)}$$

$$\omega' > C \max(*log(\omega)^{\frac{1}{2}}, \omega^{\frac{\rho+1}{2}}) \text{ (C)}$$

By a simple calculation, we can satisfy (B), (C) if;

$$\eta > \omega^5 \omega'^4 \exp(B\omega^\rho), \omega' > \omega^2 \text{ (D)}$$

and (D) if;

$$\eta > \omega'^6 \exp(B\omega'), \omega' > \omega^2 \text{ (E)}$$

Therefore, it is sufficient to have;

$$\omega' < \log(\eta)^{\frac{1}{2}}, \omega < \omega'^{\frac{1}{2}} \text{ (F)}.$$

Using (A) and condition (F), we obtain the result.  $\square$

**Theorem 0.18.** *Let  $g$  be as in Definition 0.16, let  $g_\eta$  denote its measurable extension to  $\overline{\mathcal{R}_\eta}$ , and, let  $g_{\eta,\omega}$  be the truncation of  $g_\eta$ , given by;*

$$g_{\eta,\omega} = g_\eta \chi_{[-\omega,\omega]}$$

*for a nonstandard step function  $\chi_{[-\omega,\omega]}$ , with  $\omega \in {}^*\mathcal{N} \setminus \mathcal{N}$ ,  $\eta - \omega$  infinite and  $\omega < \omega'^{\frac{1}{2}}$ . Then, with  $\hat{f}$  determined by Theorem 0.9, with  $g_{\eta,\omega}$  as the boundary condition, we have;*

$$\circ(\check{(F_{\omega'} \hat{f})})|_{st^{-1}(\mathcal{T}_{>0} \times \mathcal{R})} = st^*(H_\infty)$$

*if  $\omega' < \log(\eta)^{\frac{1}{2}}$ , and  $H_\infty$  is obtained from the classical solution  $H$  of the heat equation, with boundary condition  $g$ , given in Definition 0.16.*

*Proof.* With notation as above, we have that;

$$\check{F}_{\omega'} * f = \frac{1}{2}(\check{F}_{\omega'} * (\check{\hat{f}})) \text{ (by Theorem 0.7)}$$

$$\check{F}_{\omega'} * (\check{\hat{f}}) = \check{(2F_{\omega'} \hat{f})}$$

as, by Theorem 0.7 and Remarks 0.10, for  $a, b : \overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$ ,  $\check{(\check{a} * \check{b})} = \hat{\check{a}}\hat{\check{b}} = 2a.2b = 4ab$ ,  $(*)$ , and,  $2(\check{a} * \check{b}) = \check{(\check{a} * \check{b})} = \check{(4ab)}$ , by  $(*)$  and Theorem 0.7. Therefore;

$$\check{F}_{\omega'} * f = \check{(F_{\omega'} \hat{f})}$$

and the result follows by Theorem 0.17.  $\square$

**Remarks 0.19.** *Theorem 0.18 gives a solution to the heat equation, obtained by the following steps;*

- (i). *Truncating the transfer of the boundary data.*
- (ii). *Taking the nonstandard Fourier transform of this data and solving the resulting ODE in Theorem 0.9.*
- (iii). *Truncating the solution again.*
- (iv). *Taking the inverse nonstandard Fourier transform.*
- (v). *Specialising.*

*By straightforward results on limits in nonstandard analysis, see Theorem 2.22 of [7], it follows that the above algorithm converges for  $\{m, n, n'\}$ , with  $n < (n')^{\frac{1}{2}}$ ,  $n' < \log(m)^{\frac{1}{2}}$ , (replacing  $\{\eta, \omega, \omega'\}$  respectively), as  $m \rightarrow \infty$  (noting that, for  $\eta$  infinite,  $\eta - \log(\eta)^{\frac{1}{4}}$  is infinite). It seems likely that the algorithm is faster than current methods involving a recursion over both the space and time steps. However, this still has to be decided computationally.*

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MATHEMATICS DEPARTMENT, HARRISON BUILDING, STREATHAM CAMPUS,  
UNIVERSITY OF EXETER, NORTH PARK ROAD, EXETER, DEVON, EX4 4QF,  
UNITED KINGDOM

*E-mail address:* `tdpd201@exeter.ac.uk`